Cluster Algebras
with Applications to Arithmetic Complexity

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Abstract

This document is a short exposition of the theory of Cluster Algebras followed by an application to Arithmetic Complexity Theory, specifically Subtraction-Free Complexity.

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Contents

1 Introduction

Cluster algebras were first introduced by Sergey Fomin and Andrei Zelevinsky in 2002 with the release of [FZ01] and have further released 6 more chapters. Since the discovery, cluster algebras have seen diverse and widespread use in many areas of mathematics. Applications include: Representation theory [Kel10], combinatorics (e.g. [Kad25]), mathematical physics [Fra+14] and even arithmetic complexity [FGK16]. The latter we will touch upon in Section 4 of this exposition.

This exposition will mainly be following [FWZ24], [MS13] and [FGK16]. It is expected the reader is familiar with ring theory and basic graph theory.

2 Notations and Preliminaries

We use the notation $[n] := \{1, 2, \ldots, n\}$, and for a set I we write $\widetilde{\mathbf{x}}_I = \{x_i : i \in I\}$, where x_i will always denote an indeterminant. If I is clear from the context we may drop the subscript.

Whenever there is a tag of the form A060843 we are referring to sequence A060843 in the The On-Line Encyclopedia of Integer Sequences (OEIS), [OEI25].

Let

$$R[x_1, \ldots, x_n] = R\text{-span}\{x_1^{\mu_1} \cdots x_n^{\mu_n} : \mu_1, \ldots, \mu_n \in \mathbb{Z}_{\geq 0}\}$$

be the polynomial ring in n variables over a ring, R.

The *field of fractions*, denoted $R(x_1, \ldots, x_n)$, of $R[x_1, \ldots, x_n]$ is given by $R(x_1, \ldots, x_n) = \{\frac{a}{b} : a, b \in R[x_1, \ldots, x_n], b \neq 0\} / \sim$, where $\sim$ is the equivalence relation given by

$$\frac{a}{b} \sim \frac{c}{d} \quad \text{if} \quad ad = bc.$$

A *rational function* is an element of $R(x_1, \ldots, x_n)$.

A *Laurent polynomial* in the variables, $x_1, \ldots, x_n$, over a ring, R, is an element of the ring

$$R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = R\text{-span}\{x_1^{\mu_1} \cdots x_n^{\mu_n} : \mu_1, \ldots, \mu_n \in \mathbb{Z}\}.$$

Note that a Laurent polynomial is a rational function, but a rational function is not necessarily a Laurent polynomial. For example, $\frac{1}{1-x}$ is a rational function, but *not* a Laurent polynomial.

2.1 Schur Polynomials

Schur polynomials play a major role across mathematics. In representation theory, they are the irreducible characters of GL_n [Mac98, (6.8)] and form a $\mathbb{Z}$-basis for the ring of symmetric polynomials [Mac98, (3.2)]. In this section we define Schur polynomials in two different ways, and later, in *Section 4*, we will apply the theory of Cluster Algebras to construct an efficient 'algorithm' for computing Schur polynomials.

A *partition* of $n \in \mathbb{Z}_{\geq 0}$ is a weakly decreasing sequence of positive integers which sum to n. We denote a partition, λ, by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ with which we have that $\lambda_1 + \lambda_2 + \cdots + \lambda_d = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$.

We may visualize a partition using a *Young diagram*, which consist of boxes, arranged such that there are λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. All rows are left-aligned. See **(a)** in Figure 1.

(a)

1	2	2	5	6
2	3	6		
3	5			
5	6			

(b)

Figure 1: **(a)** A Young diagram of the partition $\lambda = (5, 3, 2, 2)$. **(b)** Example of a valid semistandard Young tableaux filled using $1, \ldots, 6$.

A *semistandard Young tableaux*, using the numbers $1, 2, \ldots, k$ and of shape λ, is an assignment of a number, $i \in [k]$, to each box of a young diagram of λ such that

- Every row is *weakly* increasing.
- Every column is *strictly* increasing.

Definition 2.1. *Given a partition, $\lambda = (\lambda_1, \ldots, \lambda_d)$, the* Schur polynomial, *$s_\lambda(x_1, \ldots, x_n)$ in k variables, is given as the sum*

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_{[k]}(\lambda)} x_1^{\mu_1} \cdots x_k^{\mu_k}$$

where μ_i is the number of i in T.

Example 2.2. *Let $\lambda = (2, 2)$. Then the Schur polynomial, s_λ in 3 variables, becomes*

$$s_\lambda = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1^2 x_3^2 + x_1 x_2 x_3^2 + x_2^2 x_3^2,$$

where each monomial corresponds to one of the semistandard Young tableaux listed below:

1	1	1	1	1	2	1	1	1	2	2	2
2	2	2	3	2	3	3	3	3	3	3	3

We now give an alternative definition of the Schur polynomials which will be useful later in Section 4.

Let $n \in \mathbb{Z}_{\geq 1}$ and consider the $n \times n$ matrix,

$$M = (M_{i,j}) = \left[\frac{x_j^{i-1}}{\prod_{a=0}^{j-1}(x_j - x_a)} \right]_{i,j=1}^{n} = \begin{bmatrix} 1 & \frac{1}{(x_2-x_1)} & \frac{1}{(x_3-x_1)(x_3-x_1)} & \cdots \\ x_1 & \frac{x_2}{(x_2-x_1)} & \frac{x_3}{(x_3-x_1)(x_3-x_1)} & \cdots \\ x_1^2 & \frac{x_2^2}{(x_2-x_1)} & \frac{x_3^2}{(x_3-x_1)(x_3-x_1)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{(2.3)}$$

which the conversant reader may recognize as a 'rescaled' Vandermonde matrix.

Let $I \subseteq [n]$ be of size k. The *chamber minor* of a matrix is defined as the determinant of the submatrix of formed by rows indexed by I and columns indexed by $[k]$.

For subset, $I \subseteq [n]$ of cardinality k, we define s_I as the chamber minor of M. I.e.

$$s_I(x_1, \ldots, x_k) := \det([M_{i,j}]_{i \in I, j \in [k]}).$$

Proposition 2.4 ([FGK16]). *Let $n \in \mathbb{Z}_{\geq 1}$ and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition satisfying $\lambda_1 + k \leq n$. Define $I(\lambda) := \{\lambda_k + 1, \lambda_{k-1} + 2, \ldots, \lambda_1 + k\}$, then the Schur polynomial, s_λ, is given by*

$$s_\lambda(x_1, \ldots, x_k) = s_{I(\lambda)}(x_1, \ldots, x_k) = \det([M_{i,j}]_{i \in I(\lambda), j \in [k]}).$$

Example 2.5. *Let $t, k \in \mathbb{Z}_{\geq 1}$ and $I = \{t+1, t+2, \ldots, t+k\}$. Then $\lambda = (\overbrace{t, t, \ldots, t}^{k})$, whose a Young diagram is a t by k rectangle. We have that*

$$s_I(x_1, \ldots, x_k) = \det \left[\frac{x_j^{i-1}}{\prod_{a=0}^{j-1}(x_j - x_a)} \right]_{\substack{t+1 \leq i \leq t+k \\ 1 \leq j \leq k}} = \frac{(x_1 \cdots x_k)^t \det[x_j^{i-1}]_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}}}{\prod_{a=0}^{j-1}(x_j - x_a)} = (x_1 \cdots x_k)^t,$$

where in the third equality we use the fact that $\det[x_j^{i-1}]_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}}$ is the Vandermonde matrix which has determinant $\prod_{a=0}^{j-1}(x_j - x_a)$ [Bäc13].

3 Cluster Algebras

In this section we give the definition of a Cluster Algebra of geometric type. We will then see the prototypical example of Pseudo-line arrangements as well as state one of the founding results; the Laurent phenomenon.

This section is inspired by [MS13] and [FWZ24].

3.1 Definition and Basic Properties

While there are more general ways of defining a cluster algebra [Mar22], in this exposition we will only be concerned with cluster algebras of *geometric type*, however, for the sake of brevity, we will simply call them a cluster algebras.

Recall that $\mathbb{Q}(u_1, \ldots, u_n)$ denotes the field of rational functions. A *cluster algebra* is a subring $C \subseteq \mathbb{Q}(u_1, \ldots, u_n)$, along with the following data:

- Combinatorial data.
- Algebraic data.

Which are manipulated by so-called *mutations*: The combinatorial data is manipulated iteratively (e.g. the local moves from Section 3.3), and the algebraic data evolves along with the combinatorial data through subtraction-free birational transformations, called *exchange relations* (e.g. (3.19) in Section 3.3).

Definition 3.1 (seed)**.** *Fix $n, k \in \mathbb{Z}_{\geq 1}$ such that $k \leq n$. An (n,k)-*seed *is a pair $(\widetilde{\boldsymbol{x}}, \widetilde{B})$ satisfying the following condition:*

1. *$\widetilde{\boldsymbol{x}} = \{x_1, \ldots, x_n\}$ have the properties that $\mathbb{Q}(u_1, \ldots, u_n) = \mathbb{Q}(x_1, \ldots, x_n)$ and $x_1, \ldots, x_n$ are algebraically independent.*
2. *$\widetilde{B} := (B_{i,j})_{i \in [n], j \in [k]} \in M_{n \times k}(\mathbb{Z})$ where the first k rows and columns are indexed by $\boldsymbol{x} = \{x_1, \ldots, x_k\}$ and the last $n-k$ rows are indexed by $\boldsymbol{c} = \{x_{k+1}, \ldots, x_n\}$.*
3. *The* principal part*, $B := (B_{i,j})_{i,j \in [k]}$, is* skew-symmetric*, meaning either $b_{i,j} = b_{j,i} = 0$ or $b_{i,j}$ is non-zero and $b_{j,i} = -b_{i,j}$, for all $i, j \in [k]$.*

Note that by (3) $b_{i,i} = 0$ for $i \in [k]$.

Remark 3.2. The above definition extends naturally to allow B to be a *skew-symmetrizable matrix*, meaning there exists a diagonal matrix D such that DB is skew-symmetric.

We use the following terminology for the seed $(\widetilde{\boldsymbol{x}}, \widetilde{B})$:

- The *cluster* is $\boldsymbol{x} = \{x_1, \ldots, x_k\}$.
- The *frozen variables* are the elements of $\boldsymbol{c} = \{x_{k+1}, \ldots, x_n\}$
- The *extended cluster* is $\widetilde{\boldsymbol{x}} = \boldsymbol{x} \cup \boldsymbol{c} = \{x_1, \ldots, x_n\}$
- The *exchange matrix* is $\widetilde{B}$.

Definition 3.3. *Two seeds $(\widetilde{\boldsymbol{x}}, \widetilde{B})$ and $(\widetilde{\boldsymbol{y}}, \widetilde{C})$ are* equivalent *if there exists a permutation, $\pi \in S_n$, acting on $[n]$, such that*

1. *$\pi(i) = i$ for all $i \in \{k+1, \ldots, n\}$.*

2. $y_i = x_{\pi(i)}$ *for all* $i \in [n]$.

3. $\widetilde{C}_{i,j} = B_{\pi(i),\pi(j)}$ *for all* $i, j \in [n]$.

We now define an important operation on the seeds called a *mutation*:

Definition 3.4 (Mutation). *Let* $\ell \in [k]$. *A* mutation, $\mu_\ell(\widetilde{\boldsymbol{x}}, \widetilde{B})$, *in direction* ℓ, *transforms a seed,* $(\widetilde{\boldsymbol{x}}, \widetilde{B})$, *into a new seed* $(\widetilde{\boldsymbol{y}}, \widetilde{C}) := \mu_\ell(\widetilde{\boldsymbol{x}}, \widetilde{B})$, *defined by*

- $\widetilde{\boldsymbol{y}} = \{y_1, \ldots, y_n\}$, *where* $y_i = x_i$ *for* $i = \ell$ *and* y_ℓ *is given by the following relation*

$$x_\ell y_\ell = \prod_{j \in [n] : B_{j,\ell} > 0} x_j^{B_{j,\ell}} + \prod_{j \in [n] : B_{j,\ell} < 0} x_j^{-B_{j,\ell}}, \tag{(3.5)}$$

  where if one of the indexing sets is empty we set the corresponding product to 1.

- $\widetilde{C} = (C_{i,j})_{i \in [n], j \in [k]}$ *is given by*

$$C_{i,j} = \begin{cases} -B_{i,j}, & \text{if } i = \ell \text{ or } j = \ell\,, \\ B_{i,j} + B_{i,\ell} B_{\ell,j}, & \text{if } B_{i,\ell} > 0 \text{ and } B_{\ell,j} > 0\,, \\ B_{i,j} - B_{i,\ell} B_{\ell,j}, & \text{if } B_{i,\ell} < 0 \text{ and } B_{\ell,j} < 0\,, \\ B_{i,j}, & \text{otherwise.} \end{cases}$$

Remark 3.6 ([FWZ24]). One can show that the following properties hold for the exchange matrix under mutation:

1. $\mu_\ell(\mu_\ell(\widetilde{B})) = \widetilde{B}$.

2. if $B_{\ell,m} = B_{m,\ell} = 0$ then $\mu_\ell(\mu_m(\widetilde{B})) = \mu_m(\mu_\ell(\widetilde{\boldsymbol{x}}))$.

Two seeds, $(\widetilde{\boldsymbol{x}}, \widetilde{B}), (\widetilde{\boldsymbol{y}}, \widetilde{C})$, are *mutation equivalent*, denoted $(\widetilde{\boldsymbol{x}}, \widetilde{B}) \sim (\widetilde{\boldsymbol{y}}, \widetilde{C})$, if there exists a sequence of mutations, $\mu_{i_1}, \ldots, \mu_{i_m}$, such that $\mu_{i_m} \circ \cdots \circ \mu_{i_1}(\widetilde{\boldsymbol{x}}, \widetilde{B}) = (\widetilde{\boldsymbol{y}}, \widetilde{C})$. By Remark 3.6 (1) we also have $\mu_{i_1} \circ \cdots \circ \mu_{i_m}(\widetilde{\boldsymbol{y}}, \widetilde{C}) = (\widetilde{\boldsymbol{x}}, \widetilde{B})$.

Definition 3.7 (Cluster Algebra). *Let* $(\widetilde{\boldsymbol{x}}, \widetilde{B})$ *be an initial* (n, k)*-seed and let* $R = \mathbb{Q}[x_{k+1}, \ldots, x_n]$ *be the polynomial ring generated by the frozen variables. Let*

$$\mathcal{S} = \{(\widetilde{\boldsymbol{y}}, \widetilde{C}) : (\widetilde{\boldsymbol{y}}, \widetilde{C}) \sim (\widetilde{\boldsymbol{x}}, \widetilde{B})\} \quad \text{and} \quad \mathcal{X} = \bigcup_{(\widetilde{\boldsymbol{x}}, \widetilde{B}) \in \mathcal{S}} \widetilde{\boldsymbol{x}}$$

The cluster algebra, $\mathcal{A}(\widetilde{\boldsymbol{x}}, \widetilde{B})$, *is defined as the* R*-subalgebra,*

$$\mathcal{A}(\widetilde{\boldsymbol{x}}, \widetilde{B}) = R[\mathcal{X}].$$

The rank *of the cluster algebra,* $\mathcal{A}(\widetilde{\boldsymbol{x}}, \widetilde{B})$, *is* k.

In words, we consider any finite sequence, $\mu_1 \circ \mu_2 \circ \cdots \circ \mu_t$, of mutations and apply it to our initial seed, $(\widetilde{\boldsymbol{x}}, \widetilde{B})$, resulting in a new seed, $(\widetilde{\boldsymbol{y}}, \widetilde{C}) = \mu_1 \circ \mu_2 \circ \cdots \mu_t(\widetilde{\boldsymbol{x}}, \widetilde{B})$. We then take the cluster $\widetilde{\boldsymbol{y}}$ and add these elements to the set of generators for $\mathcal{A}(\widetilde{\boldsymbol{x}}, \widetilde{B})$. The collection of all such clusters is $\mathcal{X}$ and so we get $\mathcal{A}(\widetilde{\boldsymbol{x}}, \widetilde{B}) = R[\mathcal{X}]$.

Example 3.8. *([FWZ24, Example 3.2.4]) Consider the initial seed $(\widetilde{\boldsymbol{x}}, \widetilde{B})$, given by $\widetilde{\boldsymbol{x}} = \{z_1, z_2\}$,*

$$\widetilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and no frozen variables.

There are only two possible mutations, μ_1, μ_2, so, by Remark 3.6 (1), the only sequences of mutations worth considering are sequences alternating between μ_1 and μ_2. This yields the following picture

$$\cdots \overset{\mu_1}{\longleftrightarrow} \overset{z_1 \quad z_0}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \overset{\mu_2}{\longleftrightarrow} \overset{z_1 \quad z_2}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \overset{\mu_1}{\longleftrightarrow} \overset{z_3 \quad z_2}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \overset{\mu_2}{\longleftrightarrow} \overset{z_3 \quad z_4}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \overset{\mu_1}{\longleftrightarrow} \cdots,$$

where $\{z_i\}_{i \in \mathbb{Z}}$ are clusters and are placed on top of their index in the corresponding exchange matrix.

The cluster variables satisfy the relation

$$z_{k+1} z_{k-1} = z_k + 1, \tag{(3.9)}$$

which we get from (3.5).

Using (3.9) to compute z_i in terms of z_1 and z_2 yields

$$z_3 = \frac{z_2 + 1}{z_1}, \quad z_4 = \frac{z_1 + z_2 + 1}{z_1 z_2}, \quad z_5 = \frac{z_1 + 1}{z_2}, \quad z_6 = z_1, \quad z_7 = z_2, \cdots$$

The sequence $(\ldots, z_{-1}, z_0, z_1, z_2, \ldots)$ is 5-periodic. There are thus 5 distinct cluster variables and the cluster algebra is

$$\mathcal{A}(\widetilde{\boldsymbol{x}}, \widetilde{B}) = \mathbb{Q}[z_1, z_2, z_3, z_4, z_5] = \mathbb{Q}\left[z_1, z_2, \frac{z_2 + 1}{z_1}, \frac{z_1 + z_2 + 1}{z_1 z_2}, \frac{z_1 + 1}{z_2}\right].$$

3.2 Cluster Algebras Represented By Quivers

If the exchange matrix is skew-symmetric, then every cluster algebra can be obtained from a quiver, along with quiver mutations, which will be defined here. This yields a nice visualization of cluster algebras:

Definition 3.10. *A* Quiver, *Q, is an oriented graph, without loops and 2-cycles, given by a set of vertices, Q_0, and a set of* arrows, *Q_1, as well as two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$, mapping an arrow to its source and target, respectively.*

A subquiver, *Q' of Q, is a quiver with vertices $Q_0' \subseteq Q_0$ and arrows $Q_1' \subseteq Q_1$.*

Construction 3.11. *A quiver Q is obtained from the seed $(\widetilde{\boldsymbol{x}}, \widetilde{B})$ by labeling the vertices by elements of the extended cluster, $\widetilde{\boldsymbol{x}}$, and for $i \in [n], j \in [k]$ we have $\max(0, \widetilde{B}_{i,j})$ many arrows $x_i \to x_j$.*

A vertex x is called *mutable* if $x \in \boldsymbol{x}$ and called *frozen* if $x \in \boldsymbol{c}$.

We will visualize mutable vertices by drawing a circle around them, and frozen vertices by drawing a box around them.

Definition 3.12. *Let x be a mutable vertex of a quiver, Q. Define the mutated quiver $\mu_k(Q)$ as having the same vertices except x which is relabeled by y given by the relation*

$$xy = \prod_{y \leftarrow z} z + \prod_{y \to z} z,$$

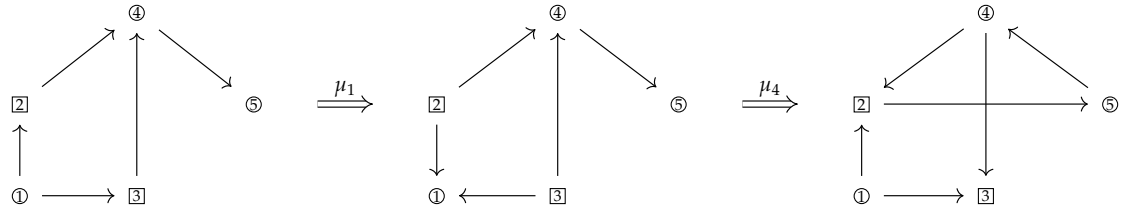
where two products are taking over edges with target y and source y, respectively.

The arrows of $\mu_k(Q)$ are given by applying the following procedure to Q:

1. *For each subquiver $i \to k \to j$ add an arrow $i \to j$, unless i and j are frozen in which case we do nothing.*
2. *Reverse all arrows incident to vertex k.*
3. *Remove all oriented 2-cycles, incrementally.*

Seed mutations and quiver mutations are related by the following proposition, the proof of which is a simple unwinding of the definitions and is therefore omitted.

Proposition 3.13. *Let Q be the quiver obtained from the seed $(\widetilde{\boldsymbol{x}}, \widetilde{B})$ and let k be a mutable vertex of Q. Then $\mu_k(Q)$ is the quiver obtained from the seed $\mu_k(\widetilde{\boldsymbol{x}}, \widetilde{B})$.*

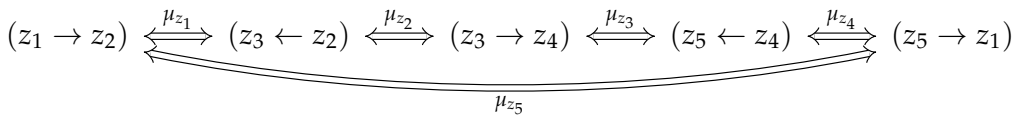


Q $\qquad$ Q' $\qquad$ Q''

Figure 2: Example of quiver mutations applied on vertex 1, followed by vertex 4. I.e. $Q' = \mu_1(Q)$ and $Q'' = \mu_4(\mu_1(Q))$.

Example 3.14. *Recall Example 3.8. Reading off $\widetilde{B} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ we get the very simple quiver $(z_1 \to z_2)$ and*

from the computations in Example 3.8, we can see that all quiver mutations are related as follows:



Note that μ_5 actually gives $(z_2 \leftarrow z_1)$ but this is equivalent to $(z_1 \to z_2)$.

For hands-on intuition, check out Bernard Keller's web app where you can construct your own quivers and apply a series of mutations to them.[1]

3.3 Pseudo-line Arrangements

In this section we present the prototypical example of a cluster algebra, which will be very useful to us in Section 4, to design an algorithm which computes Schur polynomials.

We will be some skipping some technical details and appealing more to intuition in this section. We'll be following the exposition given in, [Fom10] and [FGK16]. For a more rigorous exposition of the material see [BFZ96].

Consider the special linear group $\mathrm{SL}_n(\mathbb{C})$ and its coordinate ring, which the reader may consider as a definition:

$$\mathbb{C}[\mathrm{SL}_n(\mathbb{C})] = \mathbb{C}[x_{i,j} : i,j \in [n]]/\langle 1 - \det([x_{i,j}]_{i,j\in[n]})\rangle,$$

where $\langle 1 - \det([x_{i,j}]_{i,j\in[n]})\rangle$ is the ideal generated by the polynomial $1 - \det([x_{i,j}]_{i,j\in[n]})\rangle$.

The subgroup $B \subseteq \mathrm{SL}_n(\mathbb{C})$ of unipotent upper-triangular matrices acts on $\mathrm{SL}_n(\mathbb{C})$ by right multiplication, which induces an action $B \times \mathbb{C}[\mathrm{SL}_n(\mathbb{C})] \to \mathbb{C}[\mathrm{SL}_n(\mathbb{C})]$ defined by

$$(M, x_{i,j}) \mapsto [XM]_{i,j}$$

where $X = [x_{i,j}]_{i,j\in[n]}$. Denote the action by $M \cdot f$ for $M \in B$ and $f \in \mathbb{C}[\mathrm{SL}_n(\mathbb{C})]$.

An element $f \in \mathbb{C}[\mathrm{SL}_n(\mathbb{C})]$ is *B-invariant* if $M \cdot f = f$ for all $M \in B$.

The algebra $\mathbb{C}[\mathrm{SL}_n(\mathbb{C})]^B$ consists of polynomials in $\mathbb{C}[\mathrm{SL}_n(\mathbb{C})]$ which are invariant under the action of B:

$$\mathbb{C}[\mathrm{SL}_n(\mathbb{C})]^B = \{f \in \mathbb{C}[\mathrm{SL}_n(\mathbb{C})] : M \cdot f = f \text{ for all } M \in B\}.$$

We then have the following classic result from invariance theory:

Theorem 3.15 (Implicit in [Kos14]). *The algebra, $\mathbb{C}[\mathrm{SL}_n(\mathbb{C})]^B$, is generated*

$$\{\Delta_I : I \subsetneq [n]\}, \quad \textit{where} \quad \Delta_I = \det([x_{i,j}]_{i\in I, j\in[k]}),$$

which we call the chamber minors.

[1] See. https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/

The algebraic data of chamber minors turns out to be connected to the combinatorial data of *Pseudo-line arrangements*:

A *pseudo-line* is the graph of a continuous function $\mathbb{R}_{[-1,1]} \to \mathbb{R}^2$, where $\mathbb{R}_{[-1,1]} = \{a \in \mathbb{R} : -1 \leq a \leq 1\}$.

A *pseudo-line arrangement* is collection of n labeled pseudo-lines satisfying the following conditions:

1. Each pseudo-line is labled by a unique number from $[n]$ and has no self-intersection.
2. Each pair of pseudo-lines intersect at exactly one point.
3. No three pseudo-lines cross at the same point.

Pseudo-line arrangements are considered up to isotopy. The uninitiated reader should, for this exposition, think of pseudo-line arrangements as visualized in Figure 3.

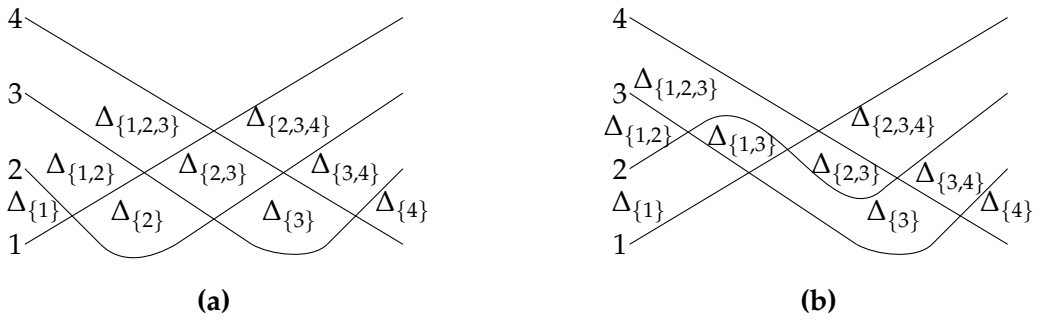


Figure 3: **(a)** a Pseudo-line arrangement of 4 pseudo-lines. **(b)** The pseudo-line arrangement yielded by a local move on $\Delta_{\{2\}}$.

A *region*, R, of a pseudo-line arrangement, P, is a connected component of the complement, $\mathbb{R}^2 - P$.

For a region, R, we define

$$I(R) = \{\text{label of } p : p \text{ is a psuedo-line passing under R}\}$$

the set of numbers associated to the set of pseudo-lines passing below R. (Again, this is defined up to isotopy).

Definition 3.16 (Local move/flip)**.** *If three pseudo-lines i, j, k cross each other and have chamber minors as shown in Figure 4, a* local move *is the moving of line j through the crossing of k and i as shown in Figure 4.*

Using the notation from Figure 4, we denote this operation by

- $\mu(\Delta_{I(R)})$ *if a chamber minor is specified.*
- $\text{flip}(i, j, k)$ *if the three pseudo-lines are specified.*

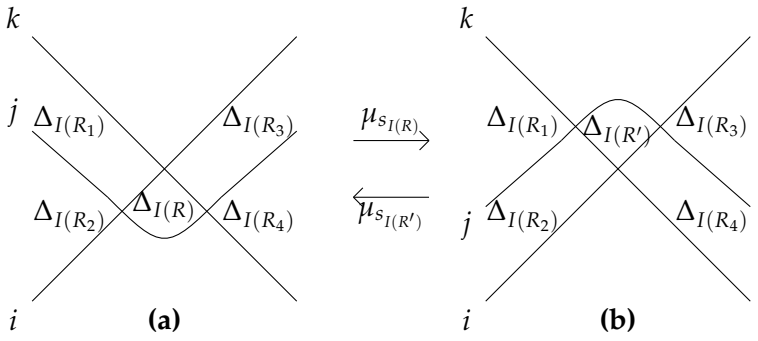


Figure 4: The local move $\mu_{s_{I(R)}}$ (or flip(i,j,k)) maps configuration **(a)** to configuration **(b)**, and vice versa for $\mu_{S_{I(R')}}$ (or flip(i,j,k)).

Fact 3.17 ([BFZ96])**.** It turns out that every pseudo-line arrangement is related to one another by a sequences of local moves.

Furthermore, the algebraic data of $\Delta_{I(R)}$ changes in a compatible way with each local move on the geometric data:

Proposition 3.18 ([Fom10])**.** *Using the notation from Figure 4, the chamber minors surrounding the local move, $\Delta_{I(R)}$ and $\Delta_{I(R')}$, are related by the equation*

$$\Delta_{I(R)}\Delta_{I(R')} = \Delta_{I(R_1)}\Delta_{I(R_3)} + \Delta_{I(R_2)}\Delta_{I(R_4)}. \tag{(3.19)}$$

This is in fact a cluster algebra, with cluster variables $\Delta_{I(R)}$ and with mutations corresponding to local moves. The exchange matrix is more complicated but can be realized as a quiver. See [Fom10] for further discussion on this example.

We finish this example with a remark which will be crucial for Section 4:

Remark 3.20. The observant reader will notice that the definition of $\Delta_{I(R)}$ is akin to that of Proposition 2.4. Indeed, we may consider the Schur polynomials $s_{I(R)}$ as the algebraic data instead of $\Delta_{I(R)}$ and all of the above still holds.

This is follows from applying the map

$$T : x_{i,j} \mapsto \frac{x_j^i}{\prod_{a=0}^{j-1}(x_j - x_a)}$$

and get from Proposition 2.4 that

$$s_I = \Delta_I(T(x_{i,j})) = \det([T(x_{i,j})]_{i\in I, j\in[k]}.$$

3.4 The Laurent Phenomenon

One of the major results for cluster algebra is that of the Laurent Phenomenon.

Theorem 3.21 (Theorem 3.3.1 [FWZ24]). *Let $\mathcal{A}$ be a cluster algebra of geometric type. Every cluster variable of $\mathcal{A}$ can be expressed as a Laurent polynomial with integer coefficients in the elements of any extended cluster.*

Which has seen widespread use. A classic example of this is that of the Somos sequences:

Example 3.22 (Somos Sequences). *Fix an integer $r \geq 2$ and consider the recurrence relation*

$$x_n x_{n+r} = \begin{cases} \sum_{i=1}^{r/2} x_{n+i} x_{n+r-i}, & \text{if } r \text{ is even,} \\ \sum_{i=1}^{(r-1)/2} x_{n+i} x_{n+r-i}, & \text{if } r \text{ is odd.} \end{cases} \tag{(3.23)}$$

Choosing $x_1 = \cdots = x_r = 1$ we get the r-Somos sequence, discovered by Micheal Somos [Som89]. For $r = 2$ or $r = 3$ this sequence is the all ones sequences, $(1, 1, 1, \ldots)$, since the right-hand side of (3.23) consists of only one term.

However, for $r = 4$ we have the relation,

$$x_n x_{n+4} = x_{n+1} x_{n+3} + x_{n+2}^2,$$

which yields the sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, \ldots \tag{A006720}$$

Since computing a new term requires division, one would expect this sequence to have fractional entries, but surprisingly it has been proven to be integral for all entries. In fact, for $r \leq 7$ the sequence is integral.

While this fact was already proven, Fomin and Zelevinsky gave a slick proof of this fact in [FZ02] by considering $x_1, \ldots, x_r$ as indeterminants and using the theory of cluster clgebras along with the Laurent phenomenon to show that each term in the sequences is a Laurent polynomial in the first r terms, and hence integral with $x_1 = \cdots = x_r = 1$.

For $r \geq 8$ it seems we eventually get fractional entries, however we were unable to find a proof of this fact. (See A030127).

4 Applications to Subtraction-Free Complexity

In this section we will see how cluster algebras can be applied to arithmetic complexity, which roughly speaking studies the hardness of computing polynomials in various models of computation. See [SY10] or [Sap21] for a survey of the field.

An important family of polynomials studied in arithmetic complexity are the Schur polynomials, as they lie on the boundary of what's computable for many models of computation. Following [FGK16] we will use Section 3.3 and the theory of cluster algebras to show that Schur polynomials can be computed easily by the 'subtraction-free complexity' model of computation.

We start by defining the most common family of models studied in arithmetic complexity.

A *binary operation* on a set S is a map $* : S \times S \to S$, $(s_1, s_2) \mapsto s_1 * s_2$.

The *in-degree* of a vertex in a directed graph is the number of edges that have the vertex as its target.

Definition 4.1 ([SY10])**.** *Let $\mathbb{F}$ be a field, $X = \{x_1, \ldots, x_n\}$ be variables and $\mathcal{O} = \{*_1, \ldots, *_t\}$ be a set of binary operations on the field of fractions $\mathbb{F}(X) = \mathbb{F}(x_1, \ldots, x_n)$.*

An arithmetic circuit*, Φ, over $\mathbb{F}$ with operations in $\mathcal{O}$ is a directed acyclic graph, $\Phi = (V, E)$, where every vertex has in-degree 0 or 2. The vertices of Φ are called* gates *and are labeled as follows:*

- *Vertices of in-degree 0 are labeled by elements of $\mathbb{F} \cup X$*
- *Vertices of in-degree 2 are labeled by elements of $\mathcal{O}$.*

A formula *is an arithmetic circuit such that the graph, Φ, is a tree.*

Each gate of an arithmetic circuit computes a rational function in a natural way: Define the function eval : $V \to \mathbb{F}(X)$ such that

- If $u \in V$ is labeled by $\alpha \in \mathbb{F} \cup X$ (i.e. of in-degree 0) then $\text{eval}(u) = \alpha$.
- If $u \in V$ is labeled by $*_i \in \mathcal{O}$ (i.e. of in-degree 2) and has children u_1 and u_2 such that $\text{eval}(u_1) = f_1$ and $\text{eval}(u_1) = f_2$ then $\text{eval}(u) = f_1 *_i f_2$

Example 4.2. *In the following two arithmetic circuits the unique sink computes $x^2 + y^2 + 2xy$:*

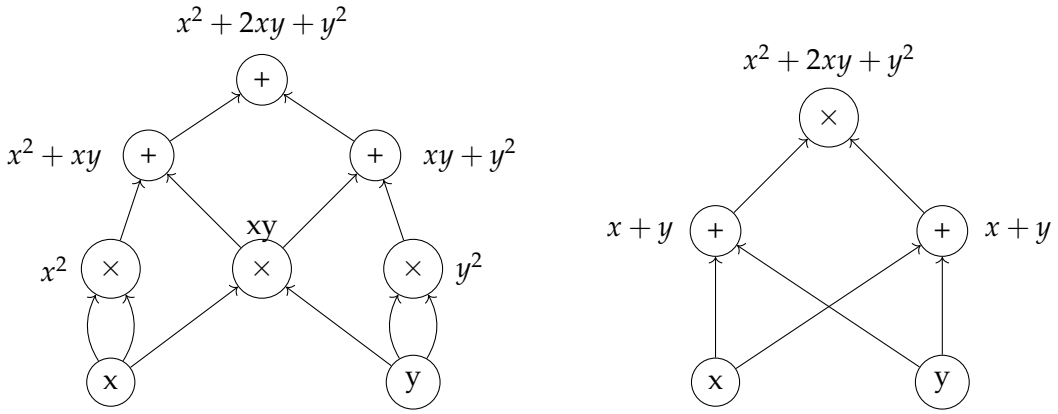


Figure 5: Two ways of computing $x^2 + y^2 + 2xy$

*We can see that **(a)** uses 6 operations while **(b)** only uses 3. Intuitively **(b)** is a more efficient circuit than **(a)**. This intuitive notion is formalized below as the* complexity measure *of arithmetic circuits.*

Definition 4.3. *Let $\Phi = (V, E)$ be an arithmetic circuit*

- *The* size *of Φ is the number of edges,* $\text{card}(E)$.

- *the* depth *of* Φ *is the length of the longest chain in* Φ.

In Figure 5, **(a)** has size 12 and depth 3 and **(b)** has size 6 and depth 2.

Remark 4.4. The size of a circuit is double the number of vertices labeled by operations since these vertices have in-degree 2. So, if a circuit has no isolated vertices the size of a circuit is asymptotically equal to the number of vertices in the circuit.

4.1 Computing Schur polynomials efficiently

A commonly considered family of arithmetic circuits are the *subtraction-free circuits*:

Definition 4.5. *A* subtraction-free circuit *is an arithmetic circuit with the following restrictions:*

1. $\mathbb{F} = \mathbb{Q}$,
2. $\mathcal{O} = \{+, \times, /\}$ *where* $+$ *is addition,* $\times$ *is multiplication and* $/$ *is division,*
3. *Every in-degree 0 vertex is labeled by* $\mathbb{Q}_{\geq 0} \cup X$.

Item 3 has the consequence that we cannot subtract and hence cannot cancel terms. This is a major limitation. Monotone circuits, which have same set-up without division, are 'exponentially weaker' than non-monotone circuits [SY10, Section 3.3].

Schur polynomials are widely believed to be hard to compute for certain models such as formulas (with $\mathcal{O} = \{+, \times\}$) [Cha+19]. Nevertheless, in this section we will show that Schur polynomials can be computed by subtraction-free circuits of size $O(n^3)$, where n is given in *Proposition* 2.4 and is a growing parameter.

Recall in Section 3.3 we can obtain any Schur polynomial from an initial seed and a series of mutations for which the Schur polynomials satisfy the (3.19), giving

$$s_{I(R)} = \frac{s_{I(R_1)}s_{I(R_3)} + s_{I(R_2)}s_{I(R_4)}}{s_{I(R')}}.$$

Notice that this equation is does not use subtraction, so to give a subtraction-free circuit of size $O(n^3)$ of the Schur polynomials, we need the following

1. Find an initial seed for which every chamber minor can be computed by a subtraction-free circuit of size $O(n^3)$.
2. Show that every Schur polynomial can be obtained from the initial seed using at most $O(n^3)$ mutations.

Theorem 4.6. *Let* $n \in \mathbb{Z}_{\geq 1}$ *be a growing parameter. Then for any* $k \in \mathbb{Z}_{\geq 1}$ *and any partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *satisfying* $\lambda_1 + k \leq n$ *there exists a subtraction-free circuit of size* $O(n^3)$ *which computes the Schur polynomial* $s_\lambda(x_1, \ldots, x_k)$.

Proof. This proof is not wholly rigorous since there are ideas left undefined. However we hope it leaves the reader with a sense of why the statement is true.

We follow the outline given just above the theorem (Theorem 4.6):

1. Consider the initial seed given by the following pseudo-line arrangement with n lines which we denote as A_n°:

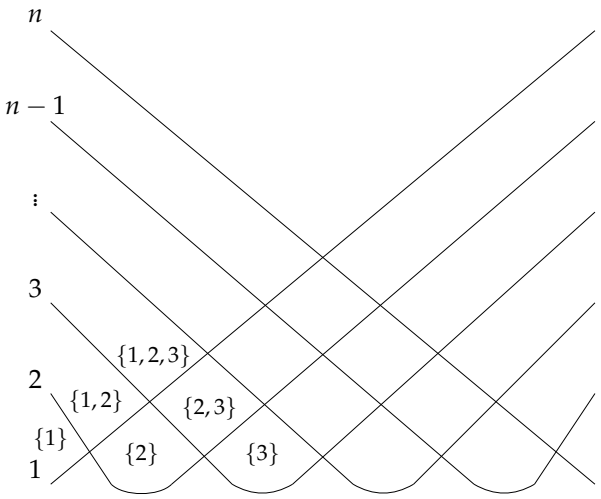


Figure 6: The pseudo-line arrangement, A°.

which has the property that every chamber minor is indexed by an interval

$$I = \{t, t+1, \ldots, t+k-1\}.$$

We saw in Example 2.5 that $s_I = (x_1 \cdots x_k)^{t-1}$. The set of monomial $\{(x_1 \cdots x_k)^{t-1} : 0 \leq t, k \leq n\}$ can be computed using $O(n^2)$ multiplications.

2. Let $I(\lambda) = \{\lambda_k + 1, \ldots, \lambda_1 + k\} = \{p_1, p_2, \ldots, p_k\} \subsetneq [n]$ with $p_1 > p_2 > \cdots > p_k$, and let $J(\lambda) = \{q_1, q_2, \ldots, q_{n-k}\} = [n] \setminus I(\lambda)$.

The following procedure yields a sequences of mutations transforming A° into a pseudo-line arrangement containing the chamber minor $s_{I(\lambda)}$:

Algorithm 1:

```
for k = 1 up to k do
    if k ∈ I(λ) then
        for j := k − 1 down to 2 do
            for i := j − 1 down to 1 do
                flip(i, j, k)
```

which uses at most $O(n^3)$ flip operatation, corresponding to $O(n^3)$ gates in the subtraction-free circuit.

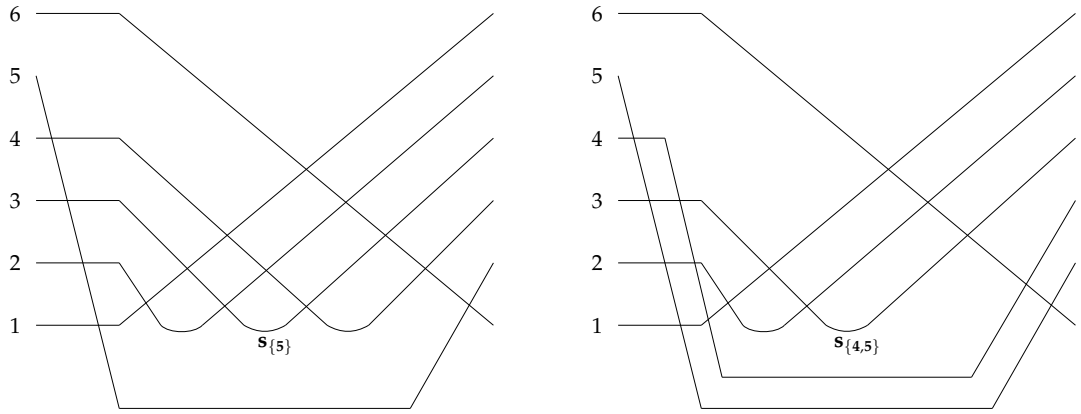


Figure 7: [FGK16]. The two iterations of k for $I(\lambda) = (4, 5)$.

It remains to justify that algorithm 1 outputs a pseudo-line arrangement containing $I(\lambda)$:

Following a line $\ell \in [n]$ left to right we have ℓ crosses the other lines exactly once in some order $(t_1, \ldots, t_i, t_j, \ldots, t_{n-1})$, where $t_k \in [n] \setminus \{\ell\}$. The key fact we use is that $\text{flip}(t_i, t_j, \ell)$ swaps t_i and t_j in the order in which ℓ crosses the lines:

$$(t_1, t_2, \ldots, t_i, t_j, \ldots, t_{n-1}) \quad \text{to} \quad (t_1, t_2, \ldots, t_j, t_i, \ldots, t_{n-1}). \tag{(4.7)}$$

This can be seen from Figure 4.

The goal is to have each line $p_r \in I(\lambda)$ cross in the order

$$(p_1, p_2, \ldots, p_{r-1},\ p_r - 1, p_r - 2, \ldots, 1,\ q_{i_1}, q_{i_2}, \ldots, q_{i_{n-p_r-r-1}}),$$

where $p_r < q_{i_j} \in J(\lambda)$, since then there is a chamber above p_k with index $I(\lambda)$. This is because there is some interval of p_k where *only* p_i have crossed p_k for all $1 \leq i < k$ and so in this interval all p_i must be below p_k with no other lines below.

At iteration $k = p_r \in I(\lambda)$, we have that initially line p_r will cross the other lines in the order (for $p_r = p_1$ this follows from our intial set-up, A_n°):

$$(p_1, p_2, \ldots, p_{r-1},\ 1, 2, \ldots, p_r - 1,\ q_{i_1}, q_{i_2}, \ldots, q_{i_{n-p_r-r-1}}),$$

where $p_r < q_{i,j} \in J(\lambda)$. Thus to get the desired form we need to reverse the subsequence, $(1, 2, \ldots, p_r - 1)$ of the order. This is exactly what the second and third for-loops do:[2]

The innermost for-loop, moves j to the $p_r - j$'th index by iteratively swapping index $t - 1$ and index t, where t is the index of j. Since we do this for all $j < k$ we get the desired order.

[2]This is essentially bubble sort on a reversed list.

□

4.2 Discussion and Open Problems

While Theorem 4.6 could be proved by more conventional ways or perhaps by just staring at (3.19) for long enough, the main advantage of the cluster algebra approach is that it provides a nice geometric and visual way of deriving circuits for certain families of polynomials, given only circuits for the cluster variables of the seed.

It was shown in [Sou25] that deciding if two quivers are mutation equivalent is an NP-hard problem.[3] However it is still open whether or not this problem is strictly harder than NP or NP-complete.[4]

Question 4.8. *Let Q_1 and Q_2 be two mutation equivalent quivers with at most n edges and vertices. Let $k = \text{poly}(n)$. Does there exist a sequences of quiver mutations, $\mu_1, ..., \mu_k$ such that $\mu_1 \circ \cdots \circ \mu_k(Q_1) = Q_2$ and $\mu_1 \circ \cdots \circ \mu_i(Q_1)$ has $\text{poly}(n)$ edges and vertices for all $i \in [k]$?*

An affirmative answer would put the quiver mutation equivalence problem in NP (and thus NP-complete). More interestingly, it would imply that if Q_1 encodes a polynomial p_1 with polynomial sized circuit then it would automatically give us an explicit circuit of polynomial size of any polynomial encoded by Q_2, under the conditions of Question 4.8.

Furthermore, an affirmative answer would also imply that polynomials such as the permanent or monomial symmetric polynomials cannot be realized as polynomially sized quivers. (Assuming VP≠VNP which is widely believed to be true). [CLS22]

It would also be interesting to see if we can obtain lower bounds for arithmetic circuits using cluster algebra theory, although there is no indication how to do this, as of writing this.

It should be noted that due to [Str73] division can be eliminated in $\{+, -, \times\}$-circuits at a polynomial cost in size. Thus any circuits found by this approach extends to circuits without division at a small cost. (But including subtraction).

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[3] I.e. there is no polynomial time algorithm for deciding if two quivers are mutation equivalent.

[4] NP-complete meaning that there is a polynomial sized proof that two quiver, Q_1 and Q_2, are mutation equivalent. For example, a proof *could* be a polynomially long series of mutations mutating Q_1 into Q_2.

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