

Tight Lower Bounds for Communication Over the Reals

**Manon Blanc, Prateek Dwivedi, Magnus Hansen,
Nutan Limaye, Meena Mahajan**

Logical foundations of complexity theory
Isaac Newton Institute - University of Cambridge

October 14, 2026

Real Communication

Alice and Bob

- Alice (A) has a vector $X_A = (X_1, \dots, X_n) \in \mathbb{R}^n$.
- Bob (B) has a vector $Y_B = (Y_1, \dots, Y_n) \in \mathbb{R}^n$.

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1 Given $P \in \mathbb{R}[X, Y]$ compute the value $P(X_A, Y_B)$.

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2 Given a set $S \subseteq \mathbb{R}^{2n}$ decide if $(X_A, Y_B) \in S$.

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We need some smoothing conditions!
(In fact, continuous is enough!)

Model for computing the value of a polynomial

Definition (Communication Complexity of a Polynomial)

For $r_1, r_2 \in \mathbb{N}$, a **polynomial-evaluation representation** of $g \in \mathbb{F}[X, Y]$ consists of polynomials $a_1, \dots, a_{r_1} \in \mathbb{F}[X]$, $b_1, \dots, b_{r_2} \in \mathbb{F}[Y]$, and a polynomial P over $\mathbb{F}$ in $r_1 + r_2$ variables such that

$$g(X, Y) = P(a_1(X), \dots, a_{r_1}(X), b_1(Y), \dots, b_{r_2}(Y)).$$

The cost of this representation is $r_1 + r_2$. The **communication complexity** of g , denoted $c(g)$, is the minimum cost of any such representation. In particular, constant polynomials have communication complexity 0.

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Remark

$$r_1 + r_2 \leq 2n.$$

Mixed Hessian and lower bounds

Definition (Mixed Hessian matrix)

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For a polynomial $g \in \mathbb{F}[X, Y]$, its $n \times n$ **mixed Hessian matrix** $H_{X|Y}(g)$ is defined by

$$[H_{X|Y}(g)]_{ij} = \frac{\partial^2 g}{\partial X_i \partial Y_j}.$$

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Lemma (Abselson)

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $g \in \mathbb{F}[X, Y]$. Suppose that

$$g(X, Y) = Q(a_1(X), \dots, a_{r_1}(X), b_1(Y), \dots, b_{r_2}(Y))$$

is a minimum-size representation of g , where $a_1, \dots, a_{r_1} \in \mathbb{F}[X]$ and $b_1, \dots, b_{r_2} \in \mathbb{F}[Y]$. Then $\text{rank } H_{X|Y}(g) \leq \min\{r_1, r_2\}$ and

$$c(g) = r_1 + r_2 \geq 2 \cdot \text{rank } H_{X|Y}(g).$$

Here the rank of a polynomial matrix is taken over $\mathbb{F}(X, Y)$.

Model for set recognition

Definition (Deterministic set-recognition protocol)

A deterministic communication protocol over $\mathbb{F}$ is a finite rooted tree, with the root at level 1, whose leaves are labeled Accept or Reject. Each internal node v at level r is assigned

- a message polynomial q_v , associated with one of the parties: either $q_v \in \mathbb{F}[X]$ or $q_v \in \mathbb{F}[Y]$; and
- a testing polynomial $P_v \in \mathbb{F}[z_1, \dots, z_r]$.

On input (X, Y) , for a v at level r , the associated party broadcasts $h_r = q_v(X)$ if the node v belongs to Alice or $h_r = q_v(Y)$ if v belongs to Bob. The outgoing edge is determined by evaluating P_v on the transcript $h = (h_1, \dots, h_r)$, where h_i are the message polynomials along a path from the root to v . Over $\mathbb{R}$, the three possible outcomes are $P_v(h) < 0$, $P_v(h) = 0$, and $P_v(h) > 0$. Over $\mathbb{C}$, the two possible outcomes are $P_v(h) = 0$ and $P_v(h) \neq 0$. The outgoing edges are labeled by these outcomes, so every input determines a unique root-to-leaf path.

The protocol **recognizes** a set $S \subseteq \mathbb{F}^n \times \mathbb{F}^n$ if this path ends at an Accept leaf exactly when $(X, Y) \in S$. The cost of the protocol is the maximum number of internal nodes on a root-to-leaf path. The **deterministic communication complexity** $\text{DCC}_{\mathbb{F}}(S)$ is the minimum cost of a protocol recognizing S . We drop the suffix when the field is clear from the context.

Example: Equality

Consider the set

$$\begin{aligned}\text{EQ}_n &= \{(X, Y) \in \mathbb{R}^{2n} \mid X = Y\} \\ &= V(X_1 - Y_1, X_2 - Y_2, \dots, X_n - Y_n) \\ &= V\left(\sum_{i=1}^n (X_i - Y_i)^2\right)\end{aligned}$$

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General recipe for $\text{PCC}_{\mathbb{R}}$ lower bounds.

$$R_F := \text{Det} \begin{pmatrix} H_{X|Y}(F) & \nabla_X F \\ (\nabla_Y F)^T & 0 \end{pmatrix}$$

Theorem

Let $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a semialgebraic set¹ and let $F \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_n]$ s.t:

- 1 F is irreducible over $\mathbb{R}$
- 2 $\partial S \cap V(F)$ has dimension $2n - 1$
- 3 $F \nmid R_F$.

Then $\text{PCC}_{\mathbb{R}}(S) \geq 2n$

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Corollary

Let $F \in \mathbb{R}[X, Y]$ be irreducible, and let $S(F) = \{(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n \mid F(X, Y) \leq 0\}$. If there exists $u \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$F(u) = 0 \quad \text{and} \quad R_F(u) \neq 0,$$

then, the probabilistic communication complexity of $S(F)$ over $\mathbb{R}$ is $\geq 2n$, i.e., $\text{PCC}_{\mathbb{R}}(S(F)) \geq 2n$.

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In particular, since F is prime it divides some $g_{i,v}$.

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Let $F \in \mathbb{F}[X, Y]$ be irreducible, and define the determinant of $(n+1) \times (n+1)$ matrix and suppose that $F \nmid R_F$. Then, for every

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Hence $g_{i,v}$ must lie at depth $2n$.

□

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