On Closure Properties of Read-Once Oblivious Algebraic Branching Programs

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Summary

Closure results

Model	**Factoring**	**Symmetric compositions**	**Powering**
VP	✓[1]	✓[5]	✓
ABP	✓[2]	✓	✓
VF	✓[3]	✓[6]	✓
VAC^0	✓[3]	✓[6]	✓
roABP	✗[7]	✗[7]	✗[7]
Sparse poly	✗[4]	✗	✗

[1] Kaltofen 1989.

[2] Sinhababu–Thierauf 2021.

[3] Bhattacharjee–Kumar–Rai–Ramanathan–Saptharishi–Saraf 2025 (arXiv:2506.23214).

[4] von zur Gathen–Kaltofen 1985.

[5] Bläser–Jindal 2019.

[6] Bhattacharjee–Kumar–Rai–Ramanathan–Saptharishi–Saraf 2025 (arXiv:2506.23220).

[7] This paper!

Notation

Throughout this talk:

- We will work over the field of complex numbers, $\mathbb{C}$.
- $\mathbf{x}$ will denote $(x_1, ..., x_n)$ for some $n \in \mathbb{N}$. (likewise for $\mathbf{y}, \mathbf{z}$).
- If $\mathbf{i} = (i_1, ..., i_n) \in \mathbb{N}^n$ then $\mathbf{x}^{\mathbf{i}}$ will denote $x_1^{i_1} \cdots x_n^{i_n}$.
- $\mathsf{coeff}_{\mathbf{x}^{\mathbf{i}}}(f)$ denotes the coefficient of monomial $\mathbf{x}^{\mathbf{i}}$ in f.

roABP - Definition & Results

Algebraic Branching Programs

Definition

An **Algebraic Branching Program (ABP)**, Φ, is a directed acyclic graph with a unique source and sink (s, t), with edges labeled by linear forms over $\mathbb{C}$.
Φ computes a polynomial f if

$$f = \sum_{p \in P} \mathrm{wt}(p),$$

where P is the set of paths $p : s \to t$ and $\mathrm{wt}(p) = \ell_1 \cdots \ell_k$, where ℓ_i are the edges of path p.

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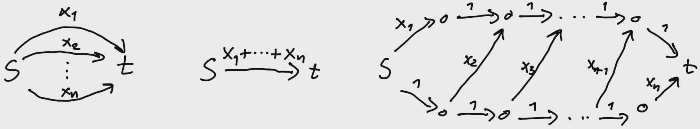
Definition

For an ABP we say that:

- The **depth** is the number of layers of the graph.
- The **width** is the maximal number of vertices of a layer throughout the graph.
- The **size** is the total number of vertices of the graph.

Examples

$$\boxed{f = x_1 + x_2 + \cdots + x_n}$$



Read-Once Oblivious Algebraic Branching Programs

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Fix a permutation $\pi : [n] \to [n]$. A **read-once oblivious algebraic branching program (roABP)** in the order π, computing $f \in \mathbb{C}[\mathbf{x}]$, is an ABP where the edges connecting layer i and $i+1$ are labeled by *univariate* polynomials over $x_{\pi(i)}$.

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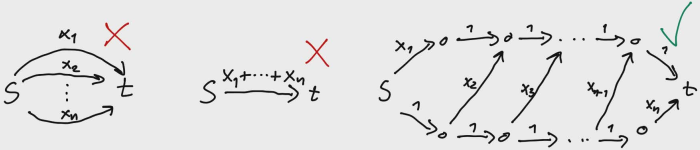
Remark 2: The notion of size is not needed: s and w denotes the size (total number of vertices) and width of an roABP then $s \leq n \cdot w$.

Examples

$$\boxed{f = x_1 + x_2 + \cdots + x_n}$$



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Definition - Fixed order width

For a fixed order $\pi : [n] \to [n]$, the width of $f \in \mathbb{C}[\mathbf{x}]$ w.r.t. π is given by

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Remark: If f has sparsity s then $\mathrm{width}_\pi(f) \leq s$.

Our Results

Non-closure under factoring

Let $n, d \in \mathbb{N}$ with $d \geq n$. Then there exists an n-variate polynomials, $f, g, h \in \mathbb{C}[\mathbf{x}]$ such that g is irreducible and $f = g \cdot h$ with

$$\mathsf{width}(f) = 2^{O(n)} \quad \text{and} \quad \mathsf{width}(g) = \mathsf{width}(f)^{\Omega(\log d)}$$

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Non-closure under symmetric compositions part 1

There exists an n-variate polynomial, f, such that

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- Why care about our results?
    - Interesting in light of recent closure results.
    - Closure results play a vital role in understanding hardness-randomness tradeoffs and the complexity of GCD and resultant.

Proof of Non-Closure under Factoring

Characterization of roABPs - Nisan Matrix [Nisan 1991]

Definition

Let $f(\mathbf{x})$ be an n-variate polynomial and let $\mathbf{y} \sqcup \mathbf{z} = \mathbf{x}$ be a variable partition.
The **Nisan Matrix**[1] w.r.t. $\mathbf{y}$ and $\mathbf{z}$ is a matrix $\mathcal{M}_{\mathbf{y},\mathbf{z}}(f)$ such that

- The rows of are indexed by monomials in $\mathbf{y}$
- The columns are indexed by monomials in $\mathbf{z}$
- If $m_{\mathbf{y}}$ and $m_{\mathbf{z}}$ are monomials in $\mathbf{y}$ and $\mathbf{z}$, then $[\mathcal{M}_{\mathbf{y},\mathbf{z}}]_{m_{\mathbf{y}},m_{\mathbf{z}}} = \mathrm{coeff}_{m_{\mathbf{y}}\cdot m_{\mathbf{z}}}(f)$

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Theorem (Nisan 1991)

*Let $f \in \mathbb{C}[\mathbf{x}]$ be n-variate and $\pi : [n] \to [n]$ a permutation.
For each $i \in [n]$ consider the variable partition given by $\mathbf{y}_i = \{\pi(x_1), \dots, \pi(x_i)\}$ and $\mathbf{z}_i = \{\pi(x_{i+1}), \dots, \pi(x_n)\}$. Then*

$$\mathrm{width}_{\pi}(f) = \max_{i \in [n]} \mathrm{rank}(\mathcal{M}_{\mathbf{y}_i,\mathbf{z}_i}(f))$$

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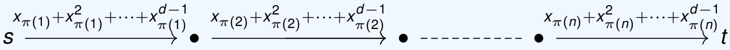

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Consider now the $2n$-variate polynomial

$$f' = f(y_1 z_1, y_2 z_2, \dots, y_n z_n) = \prod_{i \in [n]} (1 + y_i z_i + (y_i z_i)^2 + \cdots + (y_i z_i)^{d-1}).$$

and consider an order, π, such that our variables are partitioned into

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- Thus $\mathcal{M}_{\mathbf{y},\mathbf{z}}(f)$ is a *permutation matrix*.
- $\mathsf{rank}(\mathcal{M}_{\mathbf{y},\mathbf{z}}(f)) = \mathsf{card}(\{0, 1, ..., d-1\}^n) = d^n$.

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Can we lift the example for sparse polynomials to (fixed order) roABP?

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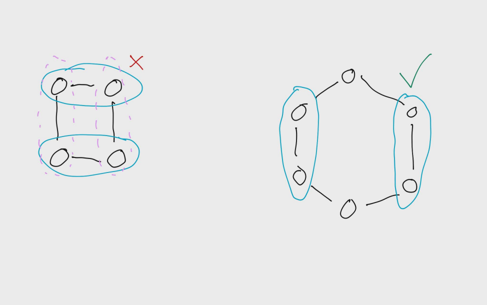

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For every $n \in \mathbb{N}$, there exists a constant degree graph $G_n = (V, E)$ on n vertices, such that the following holds:

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the graph G_n contains $\Omega(n)$ edges between S and T that form an induced matching.

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Choose d large enough that $d^{\delta} < d/3$ and apply the expander mixing lemma.

Construction

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Goal:

$$\mathrm{width}\,(g) = d^{\Omega(n)}$$

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Non-closure under symmetric compositions

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where e_k denote the **elementary symmetric polynomials**:

$$e_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

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Let S_n denote the symmetric group acting on $\mathbf{x} = \{x_1, ..., x_n\}$ by $\sigma x_i = x_{\sigma i}$ for $\sigma \in S_n$. Then

$$\mathbb{F}[\mathbf{x}]^{S_n} = \mathbb{F}[e_1, ..., e_n]$$

where e_k denote the **elementary symmetric polynomials**:

$$e_k(x_1, ..., x_n) := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

If $f \in \mathbb{F}[\mathbf{x}]^{S_n}$, we say f is **symmetric**.

Fundamental Theorem of Symmetric Polynomials

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A polynomial $f(\mathbf{x})$ is G-invariant if $f(g \cdot \mathbf{x}) = f(\mathbf{x})$ for every $g \in G$.

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Thus if $f_{\text{sym}}(\mathbf{x})$ is symmetric then there exists a polynomial $f(\mathbf{x})$ such that $f(\mathbf{e}) = f_{\text{sym}}(\mathbf{x})$.

Non-closure under symmetric compositions

Non-closure under symmetric compositions part 1

There exists an n-variate polynomial, f, such that

$$\mathrm{width}(f) = 2^{\Omega(n)} \quad \text{and} \quad \mathrm{width}(f_{\mathrm{sym}}) = O(1)$$

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Construction: Take

$$f(\mathbf{x}) = \prod_{j=0}^{k-1} \left(1 + \sum_{i=1}^{n} x_i \cdot \omega^{j \cdot i}\right) \quad \text{so} \quad f_{\mathrm{sym}}(\mathbf{x}) = \sum_{k=0}^{n} e_d(x_1^k, \ldots, x_n^k)$$

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One can show that

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Note: $f(\mathbf{x})$ is the determinant of a *circulant matrix*.

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Note: This also shows non-closure under powering

Open Problems & Thank You!

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- Does there exist a gadget ϕ such that if $f(\mathbf{x})$ has sparsity s then $f(\phi \cdot \mathbf{x})$ must have roABP complexity $\Omega(s)$? **SOLVED in the affirmative !** (But yet to be published)
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Thank you for your attention!

Questions?